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SATELLITE NAVIGATION BY TERRESTRIAL  
OCCULTATIONS OF STARS II:

CONSIDERATIONS RELATING TO  
REFRACTION AND EXTINCTION

ALI M. NAQVI

CONTRACT NO. AF33(616)-7413

AERONAUTICAL SYSTEMS DIVISION  
AIR FORCE SYSTEMS COMMAND  
UNITED STATES AIR FORCE  
WRIGHT-PATTERSON AFB, OHIO

OCTOBER 1962



GEOPHYSICS CORPORATION OF AMERICA  
BEDFORD, MASSACHUSETTS

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## FOREWORD

This study was undertaken as part of an investigation of navigation within the solar system by optical means. The objective of the investigation is to evaluate the suitability of various physical phenomena as sources of navigational information and to estimate the accuracy of navigational information obtained by various techniques. The work was supported under Contract No. AF33(616)-7413 by the Navigational and Guidance Laboratory, Wright Air Development Division, Air Research and Development Command, United States Air Force.

The present report is one of a series of three entitled:

Satellite Navigation by Terrestrial Occultations of Stars

- I. General Considerations Neglecting Atmospheric Refraction and Extinction by Ali M. Naqvi (GCA Technical Report 62-18-A).
- II. Considerations Relating to Refraction and Extinction by Ali M. Naqvi (GCA Technical Report 62-21-A).
- III. Interference Due to Brightness of Earth's Atmosphere by Rollin C. Jones and Ali M. Naqvi (GCA Technical Report 62-22-A).



## ABSTRACT

The refraction and extinction of starlight in the earth's atmosphere, including extinction due to the clouds, as applied to satellite navigation are discussed. It is concluded that the unpredictable variation in the amount and height of clouds make the observations of occultations at the earth's true edge unsuitable. A method of "virtual occultation" is described in which a change in direction, intensity, etc., of starlight due to passage through the atmosphere yields information from which the impact parameter of the observed ray can be determined and the satellite orbital elements calculated. Light extinction is preferred over refraction. A wave length band covering all or part of the region between 0.3 and 0.55  $\mu$  is recommended for extinction observations. Calculations of extinction in the atmosphere are presented and further improvements are described. An error of approximately  $\frac{1}{2}$  km in the satellite position measurement is anticipated due to unpredictable variations in atmospheric density.

## SECTION 1

### INTRODUCTION

In another report (Naqvi, 1962), hereafter referred to as Report I, a discussion of star occultations by the earth for satellite navigation, has been given, neglecting atmospheric effects. The measurements of the time of occultation of six different stars, of known coordinates, is in principle sufficient for the determination of the satellite's orbital elements. Just before an occultation the starlight must pass through the earth's atmosphere where it undergoes refraction, extinction, etc. Unfortunately the atmosphere is variable; some of the variations are fairly regular and predictable, but others are not accurately predictable. These variations can and will cause significant errors in the measurements of the occultation time. In this report we will investigate the two most important atmospheric effects and arrive at some suggestions for suitable occultation observations.

## SECTION 2

### FREQUENCY, HEIGHT AND TRANSMISSIVITY OF CLOUDS

It is well known that clouds consist of very small water droplets and, at high altitudes, of ice crystals. They are divided into several different types, whose amount and height vary with time and with geographical location in a very unpredictable manner. The distribution of the average amount and height of various cloud types for different latitudes and seasons is tabulated by London (1957). Of all the different types the cirrus clouds are found at the greatest heights. Their average amount varies from 10.0% (in winter, at 80°-90° latitude, and at an average height of 6.8 km) to 34.1% (in October at 70°-80° latitude and at an average height of 5.9 km). The average height of the base of the clouds varies from 5.3 km (in October, at 80°-90° latitude, and with an average amount of 25.0%) to 10.8 km (in Summer, at 20°-30° latitude, and with an average amount of 15.6%). However, it is not the variability of cloud characteristics but their unpredictability which is liable to cause large errors in occultation observations. It is generally believed that clouds occur up to heights of 18 km.

London (1957) has also given some data on reflectivity and absorptivity of various cloud types. Based on his data the maximum and minimum transmissivities are given in Table 2.1. The range in transmissivities is also quite large.

TABLE 2.1

## TRANSMISSIVITY OF VARIOUS CLOUD TYPES

Cloud Type	Transmissivity (%)	
	minimum	maximum
Stratus	59	73
Nimbostratus	75	83
Cumulus	77	77
Cumulonimbus	80	80
Altostratus	50	56
Cirrus	16	25

As a result of the variability of the above-mentioned cloud characteristics, the exact time of occultation becomes uncertain. For example a star may disappear behind a high altitude cloud (such as one at 18 km) before the true occultation by the earth can occur. It is possible to devise spectroscopic techniques which can distinguish an "occultation" by the clouds from an occultation by the earth. However, unless the cloud height is known, such an observation will have to be discarded, since the error in position determination can be anything up to 18 km. We, therefore, conclude that the method of occultation of stars at the earth's true edge is unsuitable.

## SECTION 3

### THE METHOD OF VIRTUAL OCCULTATIONS

In the occultation method the basic observation is the measurement of time when the ray of starlight grazes the earth's surface (Report I), i.e. passes at a known distance  $a_0$  from the earth's center, where  $a_0$  is the earth's radius. (Departures from sphericity of the earth should be considered, but they are in the nature of small corrections, and will be ignored here). The disappearance of the star at the edge of the earth provides the required signal.

It is in principle possible to determine the distance of closest approach of the (undeviated) light ray from the earth's center (the so-called impact parameter) from a change in one or more of the characteristic properties of the ray (direction, intensity, etc.) resulting from the passage of the ray through the earth's atmosphere. In practice what we propose to do is to determine, either from theory or from experiments, the precise amount of this change for a given impact parameter (say corresponding to a height of 20 km above the earth's surface, so as to avoid the clouds). The "occultation observation" would consist of the measurement of the time when the light ray has undergone the predetermined change. The accuracy of the method depends upon the accuracy with which the changes in the characteristics of the ray at various levels in the atmosphere can be

determined. We shall call this method the method of virtual occultations. The conditions for occultation, and the occultation equation derived in Report I are applicable to this method as well. In fact, in this method what we really do is to put to proper use the refraction and/or the extinction, which are otherwise sources of error in the occultation method.

In this report we shall assume spherical symmetry of the earth's atmosphere. With this assumption a ray can be uniquely characterized by its impact parameter  $r_i$  (Figure 3.1). All rays with the same impact parameter, no matter what their orientation in space, undergo the same change in their direction, intensity, etc. Note that the height  $y$  above the earth's surface, given by

$$y = r_i - a_0 \quad (3.1)$$

will also be called impact parameter, when this can be done without causing any confusion.

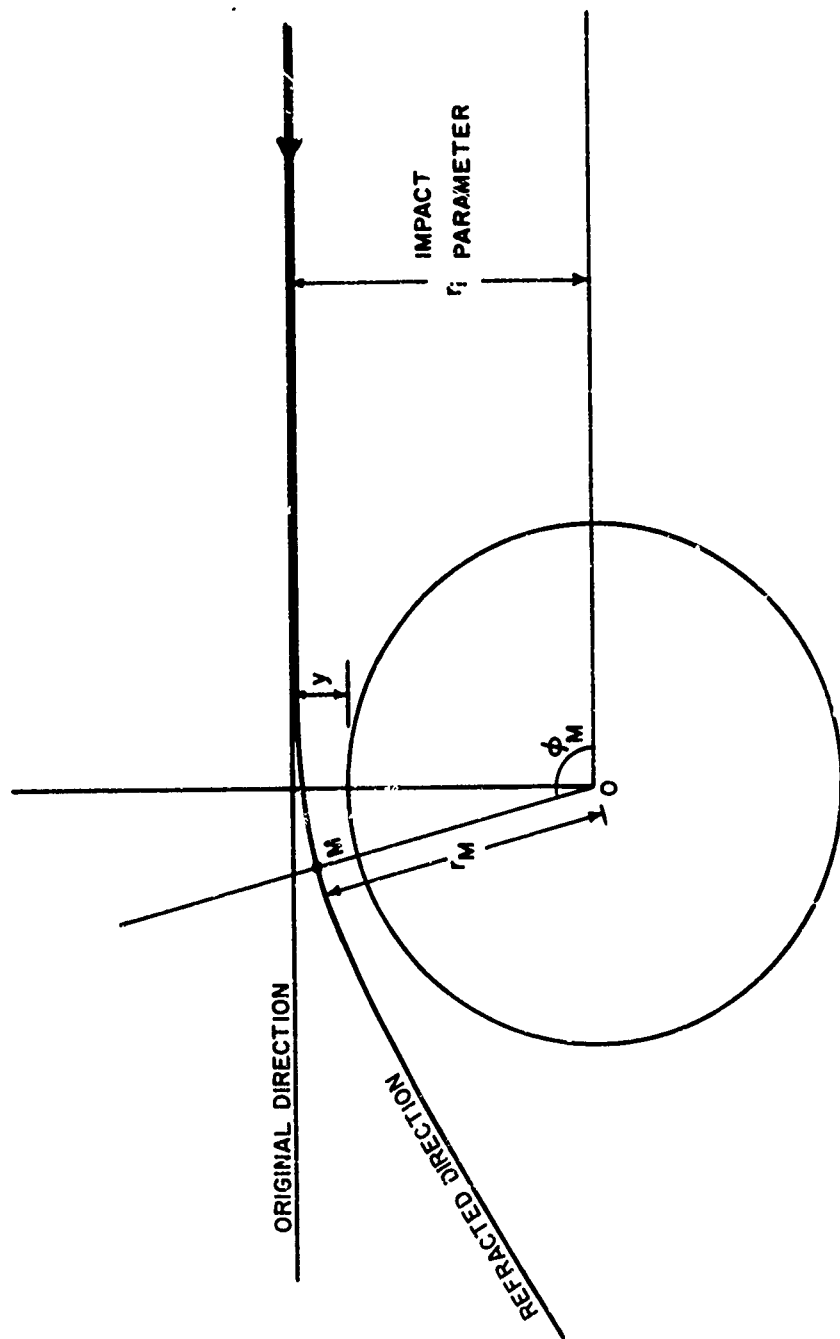


Figure 3.1. Passage of light ray through the earth's atmosphere.  
(The deviation due to refraction is shown very exaggerated.)



## SECTION 4

### EXTINCTION OF LIGHT IN THE ATMOSPHERE

#### 4.1 INTRODUCTION

There are two main causes of extinction, absorption and scattering. A third cause, known as differential refraction, arises due to the fact that a beam of starlight diverges on refraction through the atmosphere, thus causing a reduction in flux per unit area (Menzel, 1961, Section 3). However, extinction due to differential refraction is negligible at satellite distance up to many thousands of kilometers, and will not be considered here.

#### 4.2 ABSORPTION

4.2.1 Introduction. Many atmospheric constituents absorb light in the ultraviolet, visible and infrared regions. The chief absorbing molecules are  $H_2O$ ,  $N_2$ ,  $O_2$ ,  $O_3$ ,  $N_2O$ ,  $CO_2$  and  $CO$ . Although the absorption coefficients of some positive and negative ions are large, their number densities are so small as to make their contribution to total absorption negligible. Three important mechanisms for absorption are: (1) photoionization and photodissociation, (2) line absorption by atom, and (3) band absorption by molecules. Photoionization and photodissociation play their most important role in the uv region below  $0.3\mu$ . Line absorption contributes very little to the total

absorption. Band absorption, especially the vibration-rotation and the pure rotation bands, contribute primarily to absorption at long wavelength visible and the infrared regions.

4.2.2 Selection of a Suitable Wavelength Region. Prediction of the absorption of light in the earth's atmosphere requires that the distribution of absorbing molecules be either constant or its variations should be predictable. Although the distribution, as well as the total quantity, of nearly all atmospheric gases shows some variability, the gases which show very large and only partially predictable variations are water vapor, ozone and carbon dioxide.

The first water vapor bands of any consequence appear in the red region just below  $0.7 \mu$ . They become stronger towards the longer wavelengths and dominate the regions  $0.81 - 0.99 \mu$  and  $1.095 - 1.280 \mu$  and still longer wavelengths. Numerous bands of carbon dioxide occur in the region  $1.4 - 2.5 \mu$ , but none for wavelengths shorter than  $1.4 \mu$ .

Ozone has several absorption bands in the near uv and the visible region (Inn and Tanaka, 1959). The Hartley bands system, consisting of several diffuse bands, extends from  $0.320 - 0.234 \mu$  followed by strong continuous absorption to about  $0.21 \mu$  due to photodissociation of ozone. The absorption coefficient for wavelengths longer than  $0.29 \mu$  is, however, very small. Huggins bands, extending from  $0.30$  to  $0.35 \mu$ , are very weak, and the Chappuis bands, from  $0.45$  to  $0.75 \mu$ , are also very weak, particularly below  $0.55 \mu$ . Thus the wavelength

region between approximately 0.30 and 0.55  $\mu$  is practically free of water vapor, carbon dioxide and ozone absorption. It is also free of absorption due to other atmospheric gases. Goldberg (1954) states that "the bands in the entire region  $\lambda\lambda$  3200 - 5700 are so weak that atmospheric band absorption can be detected only in the spectrum of the setting sun".

Thus from the viewpoint of predictability of total absorption, any region from 0.30 to 0.55  $\mu$  appears to be the most appropriate choice. Since Rayleigh scattering coefficients (Section 4.3.2) for shorter wavelengths are much greater than for longer wavelengths, we, somewhat arbitrarily, have chosen the region from 0.3 to 0.5  $\mu$ , although any other region in the above-mentioned range, including the entire range from 0.30 to 0.55  $\mu$ , might be equally suitable.

#### 4.3 SCATTERING

4.3.1 Introduction: Mie Scattering. The two main sources of light scattering in the atmosphere are dust and aerosols (Mie scattering), and atoms and molecules (Rayleigh scattering). Dust and aerosols are permanent although minor constituents of the atmosphere. The particle sizes range from about 0.01  $\mu$  to 10  $\mu$  or even greater. Junge, Charnon and Mason (1961) and Junge (1961) have recently made direct measurements of particle density by instruments carried in balloons. Separate instruments were used for particle sizes below 0.1  $\mu$  (Aitken nuclei counter) and above 0.1  $\mu$  (collection techniques

using inertial jet impactors). For particle sizes below  $0.1 \mu$  they found concentrations of about 1000 per  $\text{cm}^3$  between 5 and 10 km decreasing to almost zero at 20 km. For particle sizes above  $0.1 \mu$  they found a maximum concentration of less than 0.1 per  $\text{cm}^3$  at 20 km, decreasing to about 0.03 per  $\text{cm}^3$  at 12 km and to about 0.01 per  $\text{cm}^3$  at 25 km. This indicates that the usual particle concentration above 20 km altitude is very small. Rozenberg (1960) has given optical evidence for scattering by aerosol near 20 km using the searchlight technique and measurements of the sky brightness in the sun's almucantar. Both methods show the existence of Mie scattering up to 20 km. Evidence for the presence of solid particles at much greater heights (85 - 100 km) is derived from observations of noctilucent clouds. Photometric and polarization measurements (Ludlam, 1957; Witt, 1960) give a particle density of  $10^{-2}$  per  $\text{cm}^3$  and an optical depth of only  $2 \times 10^{-6}$ .

We conclude the following: (1) A light ray which does not pass through the atmosphere at a height less than 20 km is suitable not only because it avoids the clouds, but also because it avoids essentially all the dust and aerosol particles. (2) For heights above 20 km and the spectral region between say  $0.3$  to  $0.5 \mu$ , the only significant cause of extinction is Rayleigh scattering.

4.3.2 Rayleigh Scattering. The energy loss per unit area due to scattering, when a monochromatic and parallel beam of radiation travels a length  $ds$  through a scattering medium is given by

$$dE_{\lambda} = -F_{\lambda} \beta_{\lambda} ds \quad (4.1)$$

where  $F_{\lambda}$  is the incident flux and  $\beta_{\lambda}$  is the monochromatic volume scattering coefficient. For Rayleigh scattering  $\beta_{\lambda}$  is given by

$$\beta_{\lambda} = \frac{32\pi^3 (\mu - 1)^2}{3N\lambda^4} \quad (4.2)$$

where  $\mu$  is the refractive index of the medium and  $N$  is the number of molecules per  $\text{cm}^3$ . When the medium consists of several species, as is the case for the earth's atmosphere, we have,

$$\beta_{\lambda} = \frac{32\pi^3}{3\lambda^4} \sum_i \frac{(\mu_i - 1)^2}{N_i} \quad (4.3)$$

where the summation extends over the various species. For the earth's atmosphere all but oxygen and nitrogen are minor constituents, and these are the only two gases which will be considered. The composition of the atmosphere varies with height  $h$  and this variation should be taken into account.

The optical thickness for vertical incidence at any height  $h$ , which we shall call the normal optical thickness, is defined by

$$\tau_{\lambda}(h) = \int_h^{\infty} \beta_{\lambda}(h) dh. \quad (4.4)$$

Deirmendjian (1955) has calculated  $\beta_{\lambda}(h)$  and  $\tau_{\lambda}(h)$  for various vertical heights and wavelengths. His results are given in Tables 4.1 and 4.2. The optical thickness even at 30 km is very small and any extinction at a height above 100 km can be neglected.

The optical thickness of the earth's atmosphere for rays for which the impact parameter is greater than the earth's radius, can be calculated very easily. Consider Figure 4.1, in which the deviation due to atmospheric refraction of the ray is neglected. This is particularly justified for rays which do not pass through the lower atmosphere.

Consider a rectangular coordinate system  $xy$ , whose  $x$  axis is along the direction of the ray and whose origin  $O$  is the point at which the light ray grazes the earth's surface. The optical thickness at point  $P$  along the ray is defined by

$$\tau_{\lambda}(x, y) = \int_h^{\infty} \beta_{\lambda}(x, y) dx. \quad (4.5)$$

The optical thickness for various values of impact parameter  $y$  and distance  $x$  have been calculated and they are plotted in Figures 4.2 and 4.3. The largest value of  $x$  used in the numerical integration of

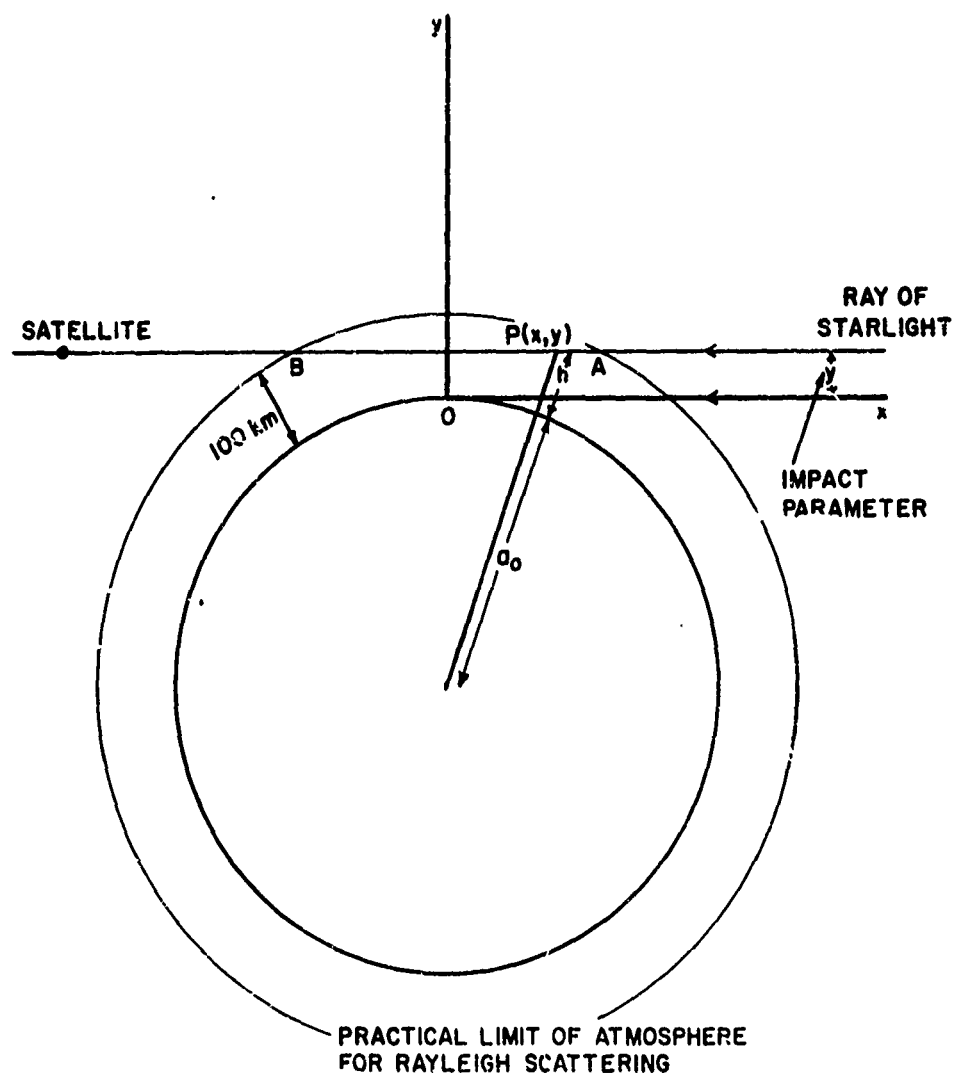


Figure 4.1. Path of light ray in earth's atmosphere. Practically all the extinction occurs between points A and B.

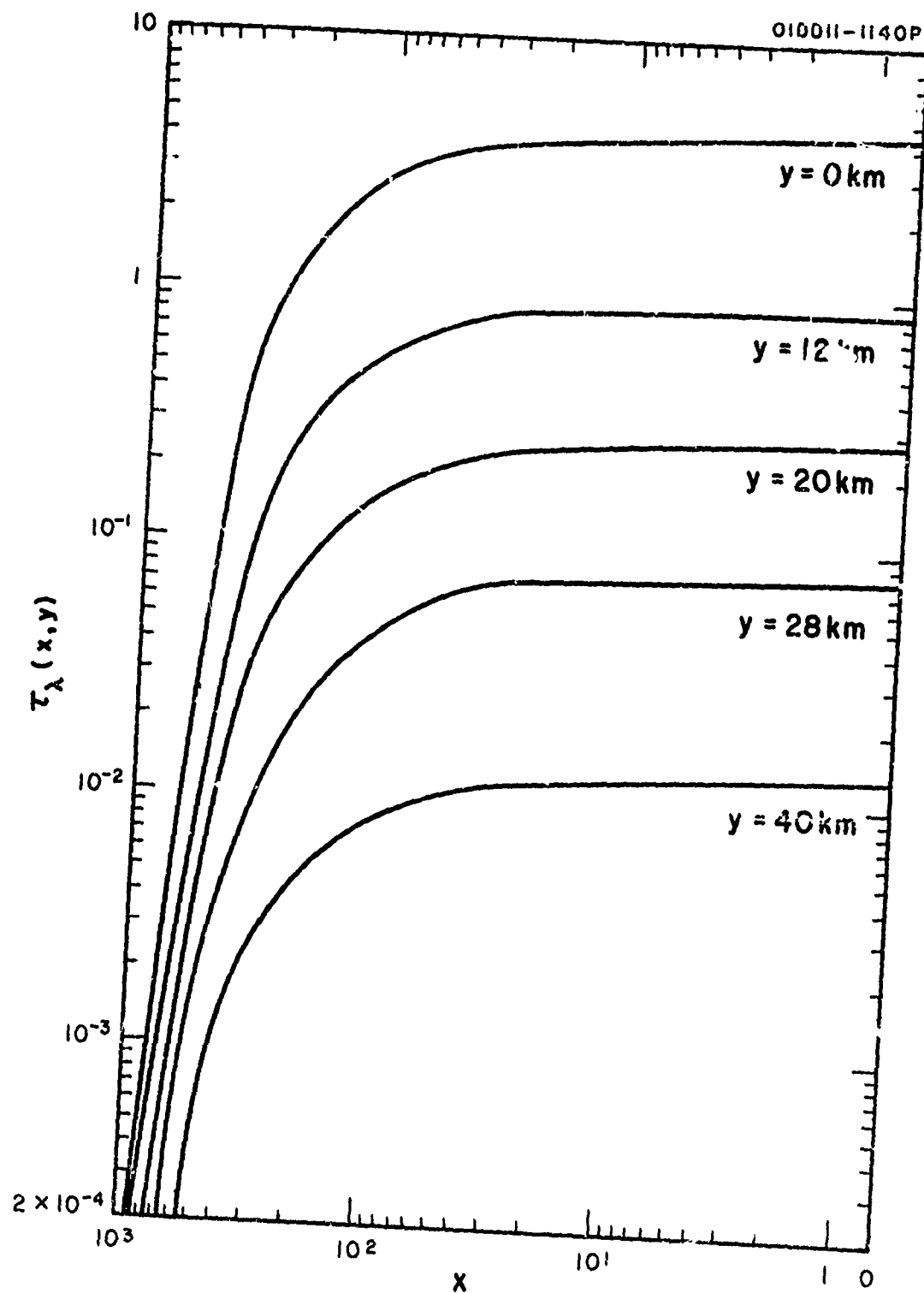


Figure 4.2. Optical thickness  $\tau_\lambda(x, y)$  of the atmosphere for various rays at a wavelength of  $0.52\mu$ .



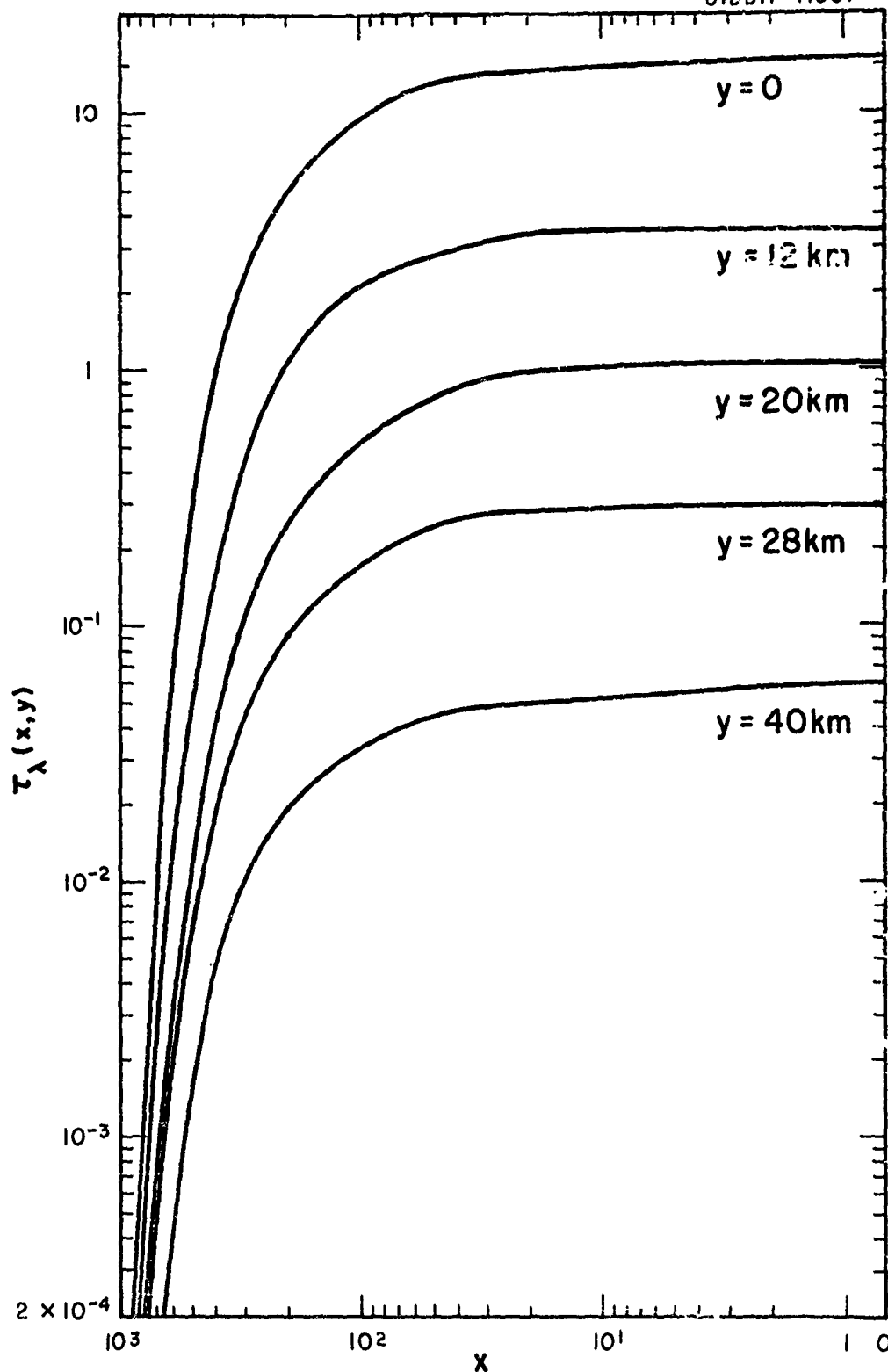


Figure 4.3. Optical thickness  $\tau_\lambda(x, y)$  of the atmosphere for various rays at a wavelength of  $0.375\mu$ .

Equation (4.5) corresponds to  $h = 100$  km.

Due to symmetry around the  $y$  axis, the optical thickness values for only the positive values of  $x$  have been calculated. The total optical thickness of the atmosphere along a ray, whose impact parameter is  $y$ , is given by

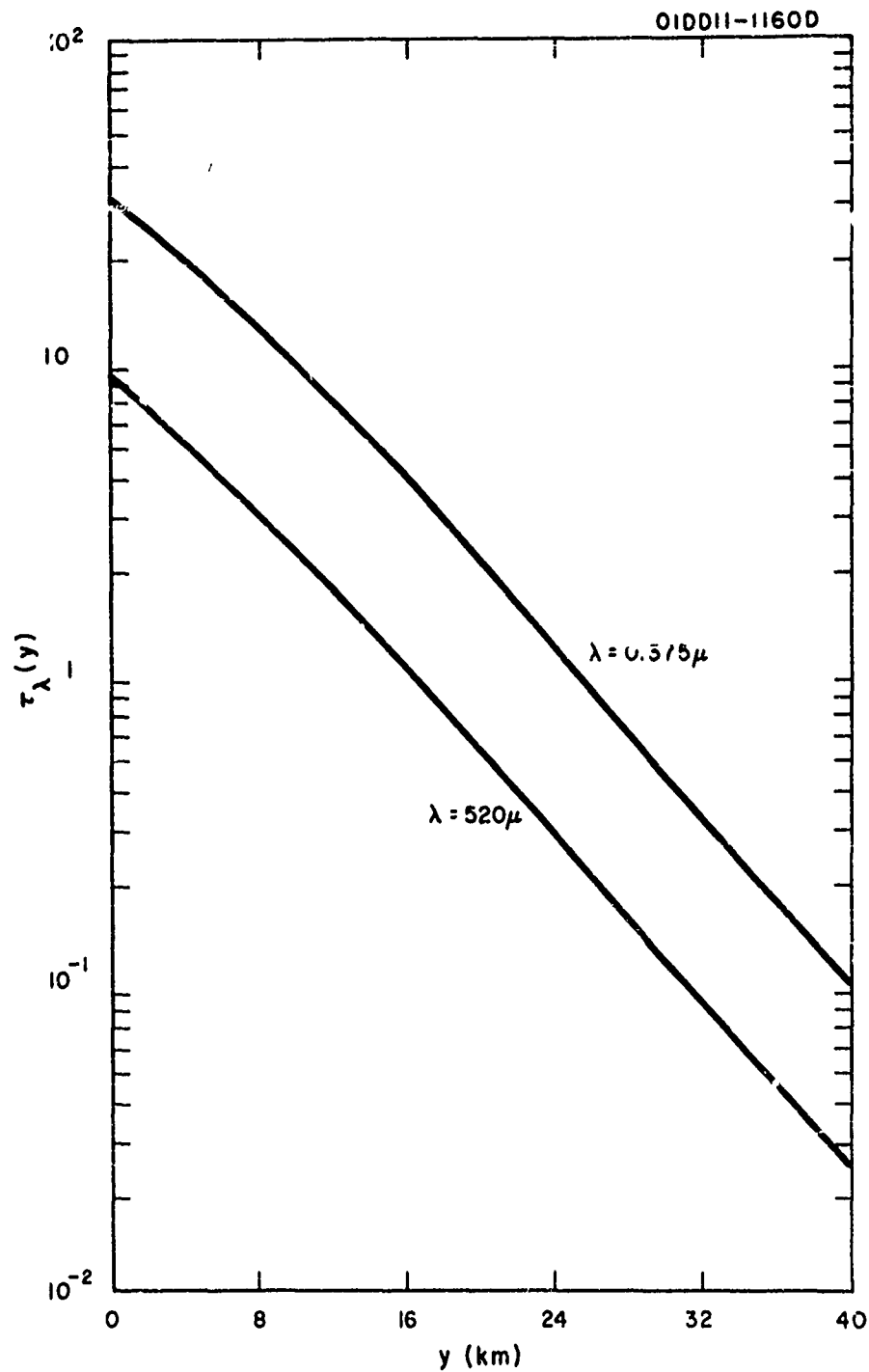
$$\tau_{\lambda}(y) = 2 \tau_{\lambda}(0, y). \quad (4.6)$$

$\tau_{\lambda}(y)$  is plotted against  $y$  in Figure 4.4.

For a satellite whose altitude is more than 100 km, the total optical thickness of the atmosphere along a given ray is given by the above equation; it does not depend upon the actual satellite height, since extinction above a height of 100 km is negligible. The extinction suffered by a ray which traverses the entire atmosphere, is therefore a function of the impact parameter  $y$  only. For an optically thin atmosphere it is given by

$$\text{Extinction} = 1 - \exp \left[ -\tau_{\lambda}(y) \right]. \quad (4.7)$$

The percentage extinction for various impact parameters has been calculated. It is plotted in Figure 4.5. Note that the atmosphere is not optically thin for impact parameters less than 25 km and 16 km for wavelengths of  $0.375\mu$  and  $0.520\mu$  respectively. If the reinforcement of light intensity due to multiple scattering is taken into account, the extinction, as given in Figure 4.5, would be found to be overestimated. However, for impact parameters up to 20 km the multiple scattering effect is not large. Corresponding to an impact



4.4. Optical thickness  $\tau_\lambda(y)$  of the entire atmosphere for various impact parameters and two wavelengths.

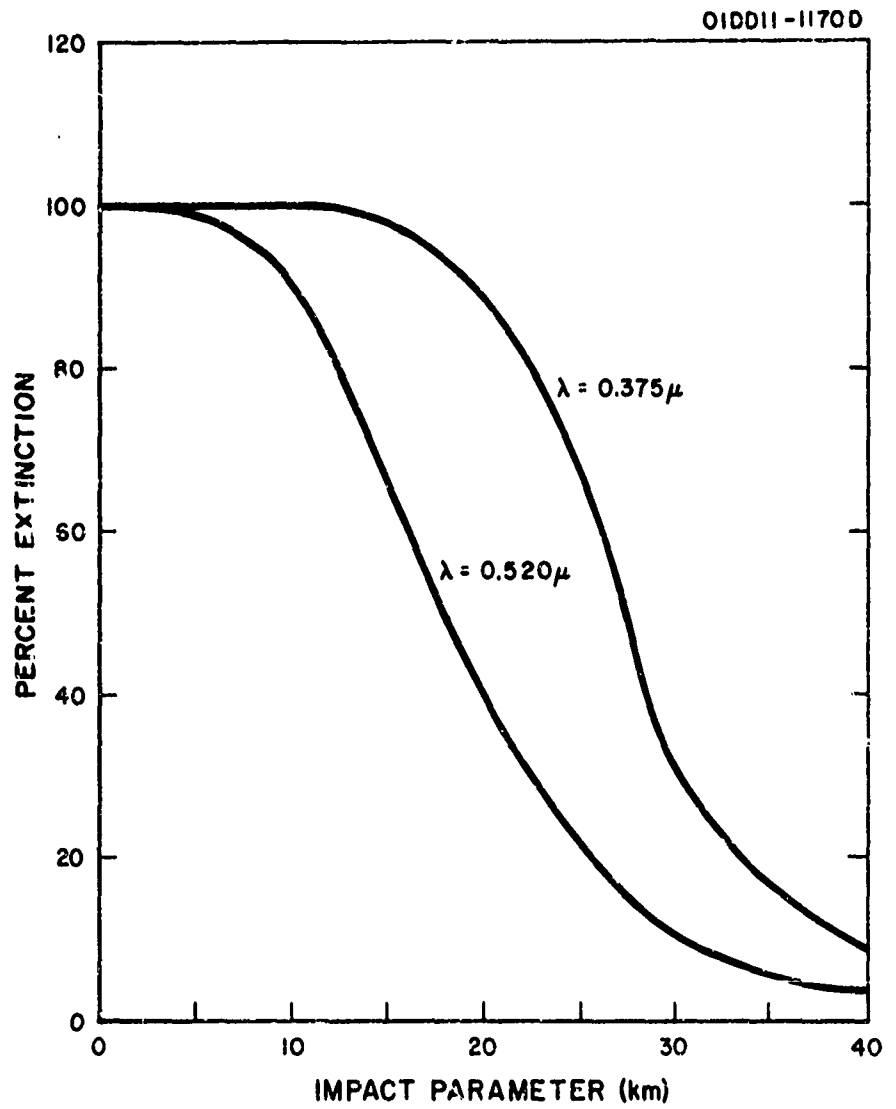


Figure 4.5. Extinction due to Rayleigh scattering for various impact parameters and two wavelengths.

parameter of 20 km, the extinction is 40% at  $\lambda = 0.520 \mu$  and 88% at  $\lambda = 0.375 \mu$ . The total extinction for any star of given spectral intensity distribution can be calculated in a similar manner. It will vary with the spectral characteristics of a star, but its calculation for individual stars is beyond the scope of the present report. The values of extinction for the above-mentioned impact parameter and for the wavelength region 0.3 to 0.5  $\mu$  are expected to lie between 60 and 80%.

It should be emphasized that the present investigation is exploratory in scope. Several improvements are possible, and should be made to the calculations presented here. They are:

1. Departures from the assumed spherical symmetry of the atmosphere should be taken into account. Due to lack of spherical symmetry the extinction will depend upon the orientation of the ray with respect to, say, the earth's equator.
2. The small contribution to extinction due to the true absorption by atmospheric gases should be calculated.
3. The model atmosphere used by Deirmendjian is based upon the Rand model of the atmosphere (Grimminger, 1948) and rocket data of Havens, Koll and La Gow (1952). Since more accurate data is now available, new calculation based upon the most accurate available data would be desirable.
4. Deirmendjian considered only two constituents of the atmosphere, nitrogen and oxygen. The contribution of other gases, although small, should be investigated.

TABLE 4.1

THE VOLUME EXTINGUISHING COEFFICIENT AS A FUNCTION OF  
HEIGHT AND WAVELENGTH IN THE EARTH'S MOLECULAR ATMOSPHERE.

(after Deirmendjian, 1955)

h(km)	$\beta_{\lambda}(h)$ in $\text{cm}^{-1}$	
	$\lambda = 375 \text{ m}\mu$	$\lambda = 520 \text{ m}\mu$
0	$5,365 \times 10^{-7}$	$1,395 \times 10^{-7}$
5	3,203	0,833
10	1,8135	0,4715
20	0,395	0,103
30	$8,42 \times 10^{-9}$	$2,19 \times 10^{-9}$
40	1,63	0,424
60	$1,77 \times 10^{-10}$	$4,61 \times 10^{-11}$
80	$1,20 \times 10^{-11}$	$3,11 \times 10^{-12}$
100	$3,76 \times 10^{-13}$	$9,78 \times 10^{-14}$

TABLE 4.2

THE COMPUTED OPTICAL THICKNESS AS A FUNCTION OF  
WAVELENGTH AND HEIGHT IN THE EARTH'S MOLECULAR ATMOSPHERE.  
(after Deirmendjian, 1955)

h(km)	$\tau_{\lambda}(h)$	
	$\lambda = 375 \text{ m}\mu$	$\lambda = 520 \text{ m}\mu$
0	0,4542	0,1181
2	0,3566	0,0927
4	0,2770	0,0720
6	0,2127	0,0553
8	0,1608	0,0418
10	0,1196	0,0311
12	0,0979	0,02285
16	0,0473	0,0123
20	0,0255	0,00663
24	0,0138	0,00358
30	0,00546	0,00142

5. Seasonal and latitudinal changes of the atmospheric density profile are well-known. Recently Alfuth, Dickey and Alsobrook (1960, 1961) have attempted to give a standard atmospheric density profile for various latitudes and months. For any accurate predictions of the atmospheric extinction it will be necessary to take these variations into account.

6 The classical scattering theory due to Rayleigh breaks down for wavelengths equal to or smaller than the resonance wavelengths of the scattering atoms and molecules. This is likely to change the scattering cross-sections for ultraviolet radiation (Moses, 1962). Further calculations of quantum mechanical scattering cross-sections are required.

7 Integrated atmospheric extinction in a given wavelength region should be calculated for a number of selected bright stars.

All of the improvements outlined above can be incorporated into future calculations without any serious difficulty, though not without a considerable effort. With the incorporation of these improvements the major source of error remains the unpredictable temporal and spatial fluctuations of the atmospheric density, even with the seasonal and the latitudinal effects incorporated

In Figures 4.6 to 4.9, we reproduce four of the several curves by Alfuth, Dickey and Alsobrook (1960). All these curves are for one location (International Falls, Minnesota) for the months shown



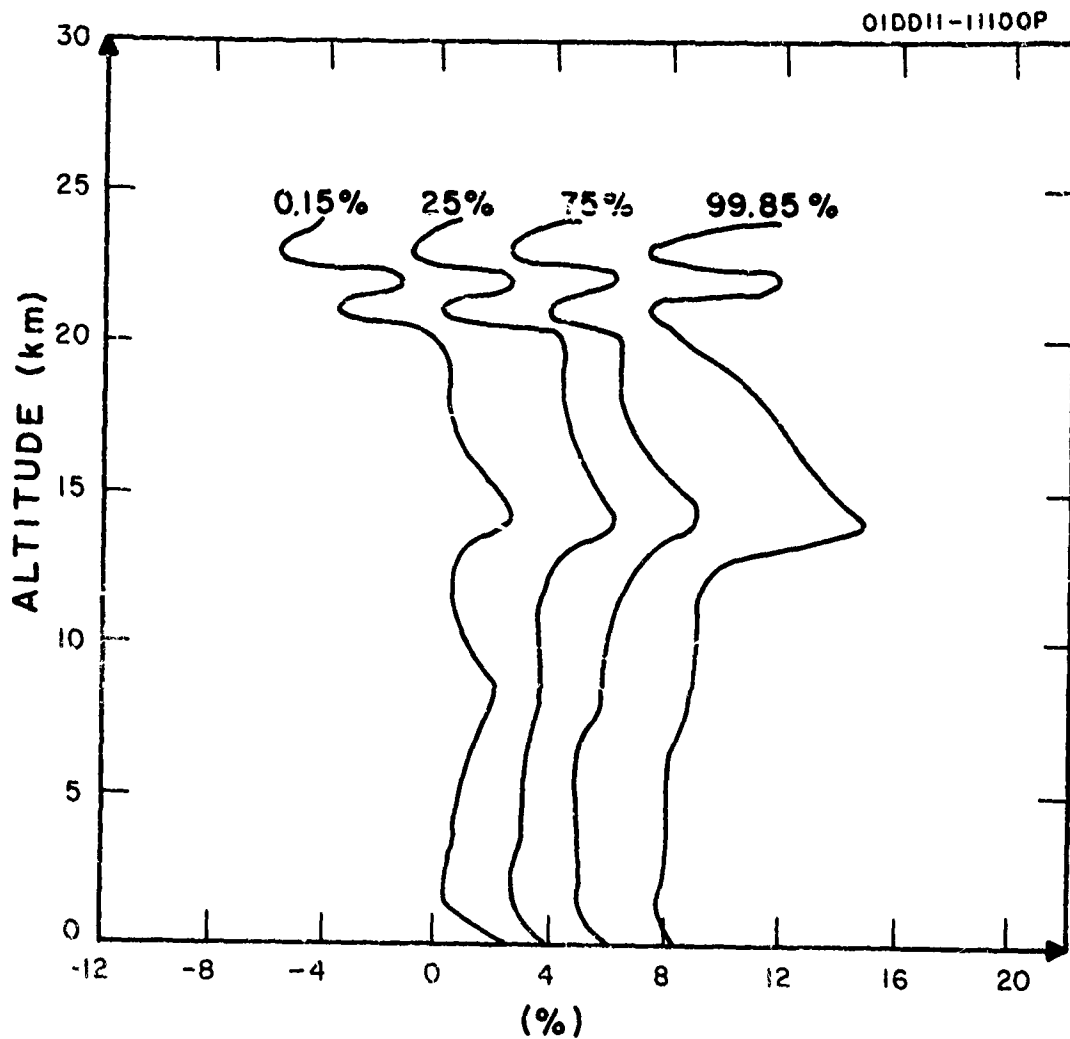


Figure 4.6. Percentage deviations of the actual atmospheric density from the standard density for four values of the cumulative percentage frequency (0.15%, 25%, 75% and 99.85%)  
Place: International Falls, Minnesota; Month, January  
(after Alfuth et al.)

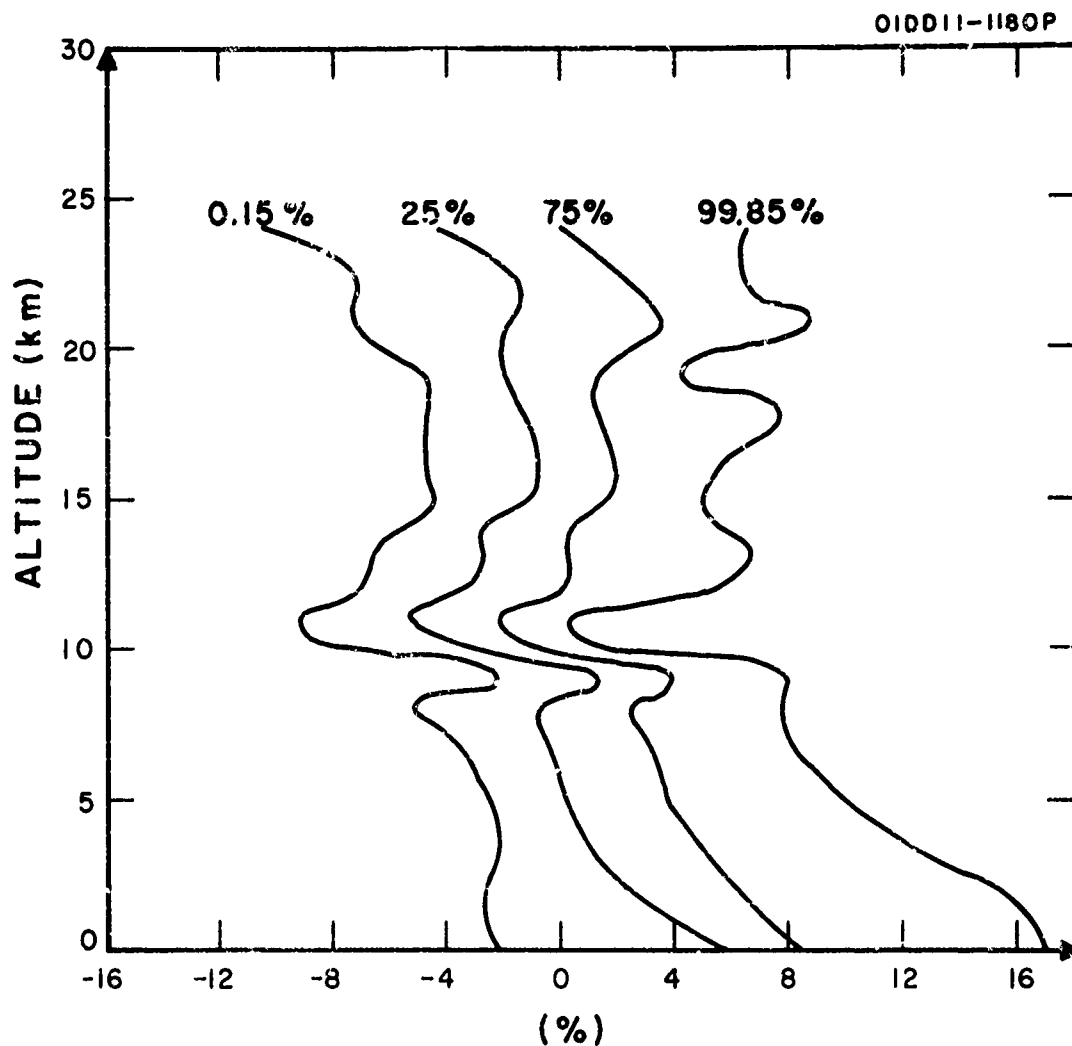


Figure 4.7. Percentage deviations of the actual atmospheric density from the standard density for four values of the cumulative percentage frequency (0.15%, 25%, 75% and 99.85%)  
 Place: International Falls, Minnesota, Month, April  
 (after Alfuth et al.)

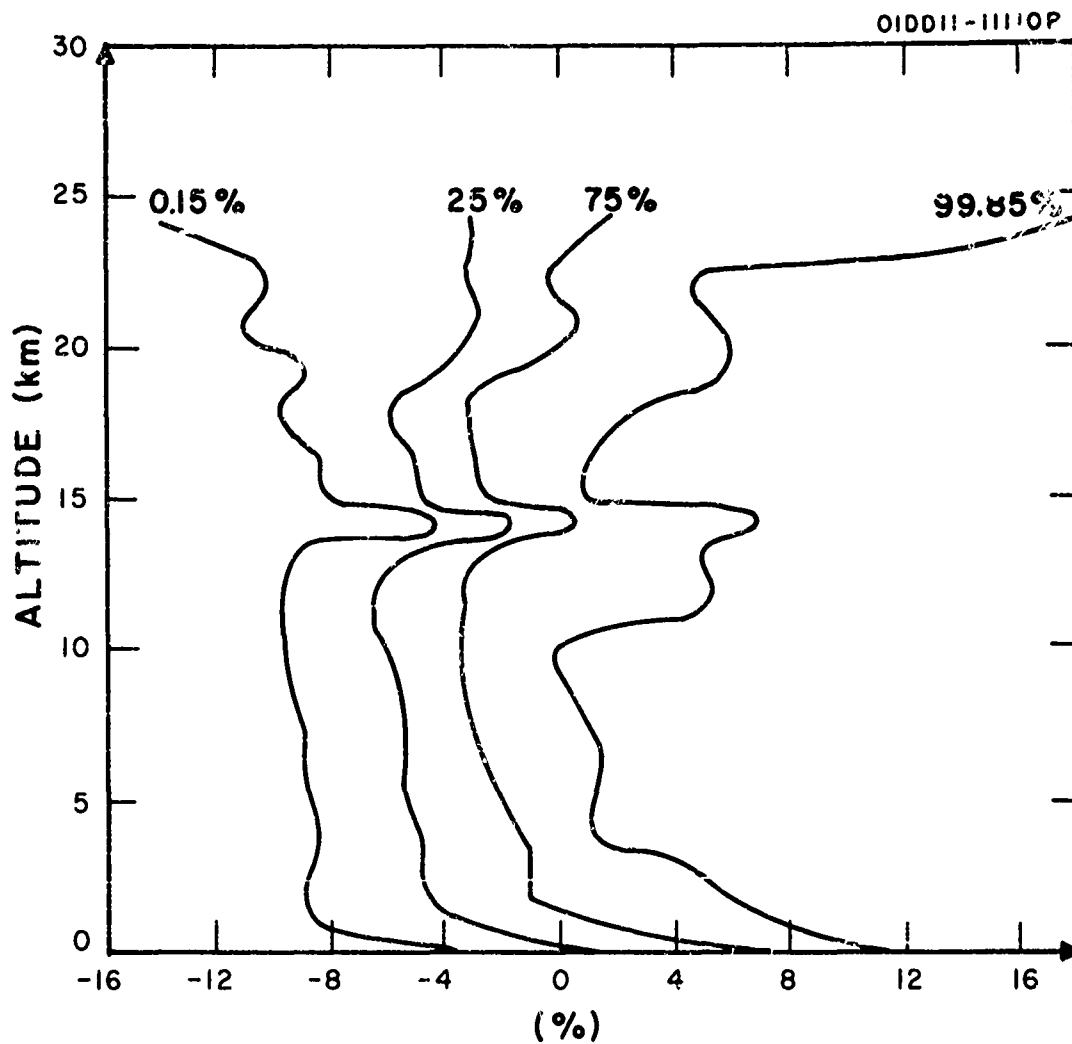


Figure 4.8. Percentage deviations of the actual atmospheric density from the standard density for four values of the cumulative percentage frequency (0.15%, 25%, 75% and 99.85%)  
 Place: International Falls, Minnesota; Month, July  
 (after Alfuth et al.)

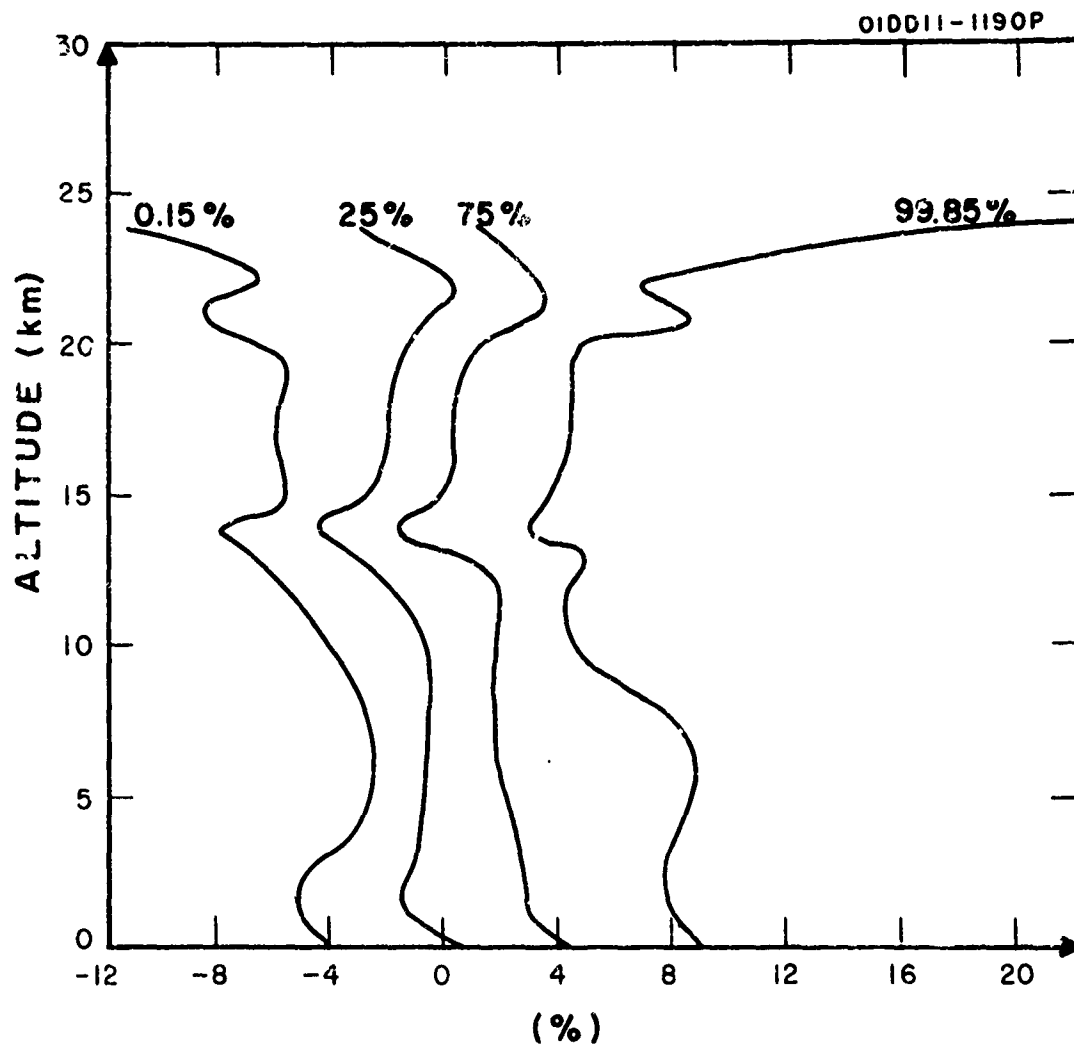


Figure 4.9. Percentage deviations of the actual atmospheric density from the standard density for four values of the cumulative percentage frequency (0.15%, 25%, 75% and 99.85%)  
Place: International Falls, Minnesota; Month, October  
(after Alfuth et al.)

The abscissae are the percentage deviation of the actual density from the standard atmospheric density (expressed as a polynomial) for four values of the cumulative percentage frequency (0.15%, 25%, 75% and 99.85%). For example, from Figure 4.6, which is from the data for International Falls, Minnesota for the month of January, the following information can be obtained for 24 km altitude. The measured density, which was such that 0.15%, 25%, 75% and 99.85% of all the measurements had values below this density, differed from the standard density by -10%, 0.4%, 5.2% and 21.6% respectively. Note that the negative sign implies that the standard density is greater than the measured density corresponding to a given cumulative percentage frequency. A study of these figures reveals that the deviations from the standard density values are quite frequently large.

Although a minimum of six occultation time measurements are sufficient for orbit determinations, many more observations seem desirable. The least square method could then be applied and erroneous observations could either be discarded or their effect on the solution of the orbital elements greatly minimized. In any case, errors of about 5% in the density at altitudes between 20 and 30 km will be very difficult to avoid. It is obvious that for rays whose impact parameter  $y$  is 20 km, almost all the extinction occurs at the altitude region mentioned above. Since the optical thickness is roughly proportional to the density, this corresponds to errors of about 5% in the calculations of optical thickness. For a ray of impact parameter

20 km,  $\tau_\lambda$  is approximately equal to 2 (Figure 4.3) and the error  $\Delta\tau_\lambda$  is approximately 0.1. Taking the derivative of Equation (4.7), the error in extinction is found to be,

$$\begin{aligned}\Delta (\text{Extinction}) &= (1 - \text{Extinction}) \Delta\tau_\lambda \\ &= 0.02\end{aligned}$$

for an extinction of 80%. From Figure (4.5), one can see that for rays with impact parameter of about 20 km, extinction is a linear function of the impact parameter  $y$ . Therefore

$$\begin{aligned}\Delta y &= y \frac{\Delta E}{E} \\ &= 0.5 \text{ km.}\end{aligned}$$

The error in the calculated position of the satellite will generally amount to less than  $\frac{1}{2}$  km.

## SECTION 5

### REFRACTION

An investigation of refraction in a planetary atmosphere, with application to occultations, has been described in another Geophysics Corporation of America Technical Report (Menzel, 1961). For details regarding refraction, the reader is referred to that report.

As in the previous sections, we again assume a spherically symmetric atmosphere. The generalized form of Snell's law is

$$\mu r \sin i = C = r_1 \quad (5.1)$$

where  $r$  is the radius vector of a point along the path of the ray,  $\mu$  and  $i$  are the refractive index of the medium and the angle of incidence of the ray at this point. The constant  $C$  is shown in the Appendix to be equal to the impact parameter  $r_1$  of the ray, provided  $\mu$  is unity at an infinite distance from the earth. Note that at some point  $M$  for which  $i$  is  $\frac{\pi}{2}$ ,

$$\mu_M r_M = r_1 \quad (5.2)$$

The differential equation of a light ray for a spherically symmetric atmosphere is

$$\left(\frac{dr}{d\phi}\right)^2 = r^2 \left( \frac{\mu^2 r^2}{r_1^2} - 1 \right) \quad (5.3)$$

$\frac{dr}{d\phi}$  is zero at the point where  $r$  equals  $\frac{r_1}{\mu}$ . It is shown in the Appendix that M is the point of closest approach, and that the path of the ray is symmetrical on both sides of M (Figure 3.1).

If the polar angle at M is denoted by  $\phi_M$ , the deviation suffered by the ray in going from infinity to M is  $\left(\phi_M - \frac{\pi}{2}\right)$ . The total deviation suffered by a ray in passing through the entire atmosphere is

$$\delta = 2\phi_M - \pi \quad (5.4)$$

For a ray grazing the earth's surface, and passing through the entire atmosphere, the deviation  $\delta$  is approximately  $1^{\circ}13'$  (twice the so-called horizontal refraction; see Chauvenet, 1891 and Menzel, 1961). The deviation at first decreases rapidly with the impact parameter, and then levels off as the refractive index rapidly approaches unity. The precision of the angle measurements using satellite borne instruments is a few minutes of arc only, and it is expected that the method using extinction of light, discussed in the previous section will give greater accuracy in satellite navigation.



## APPENDIX

### SOME GENERAL THEOREMS FOR ATMOSPHERIC REFRACTION

The following theorems hold for a spherically symmetric planetary atmosphere, independent of the manner in which the refractive index varies with the radius vector, provided it does not become discontinuous anywhere and approaches unity (for empty space) at an infinite distance. Although these theorems are intuitively obvious, and perhaps are known to others, the present author has not seen them stated in literature. For want of reference, they are stated below and elementary proofs are given.

Theorem 1. The constant in Snell's law for a ray of light

$$\mu r \sin i = C$$

equals the impact parameter  $r_1$  of the ray.

Proof: From Figure A.1 it is easy to see that

$$\lim_{r \rightarrow \infty} r \sin i = r_1$$

Since  $\mu$  is assumed to approach unity as  $r \rightarrow \infty$ ,

$$\mu r \sin i = C = r_1$$

Theorem 2. If at some point M on the path of the refracted ray  $\frac{dr}{d\phi}$  is zero, the radial distance  $r_M$  of this point is a minimum. For each refracted ray, there is one and only one point with this property.

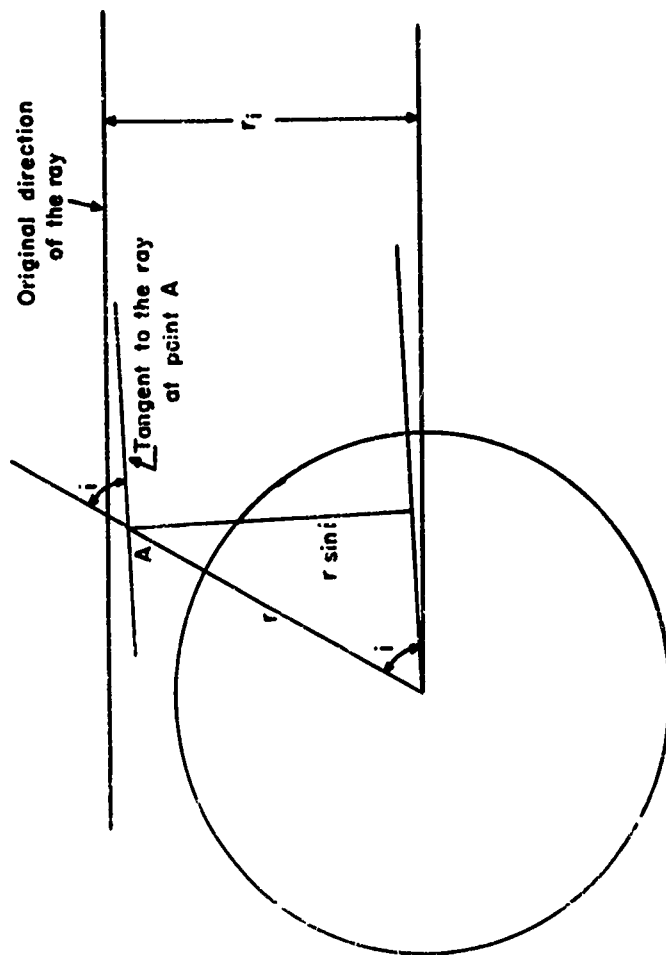


Figure A-1.

Proof: Taking derivatives of the differential equation of the light ray, (Equation 5.3 ), we get

$$\frac{d^2 r}{d\phi^2} = r \left[ \frac{\mu^2 r^2}{r_i^2} - 1 \right] + \frac{r^3 \mu^2}{r_i^2} .$$

Since at point M,

$$\left( \frac{dr}{d\phi} \right)_M = 0 ,$$

from Equation (5.3 ) it follows that

$$\mu_M^2 r_M^2 = r_i^2 ,$$

and

$$\left( \frac{d^2 r}{d\phi^2} \right)_M = r_M .$$

Since  $r_M$  is always positive, it represents the minimum, and never the maximum, distance. Furthermore, since there is no real maximum of distance (except at  $\infty$ ), there can be, at most, one minimum value of the distance. Again since  $r$  goes from infinity, through a set of finite values, to infinity again, there must be at least one minimum distance.

Theorem 3. The path of a light ray is symmetrical on both sides of the point at which  $\frac{dr}{d\phi}$  is zero.

Proof: If  $i$  is the angle of incidence of a ray going from right to left at some point A, the angle of incidence when the same ray is

going from left to right is  $\pi - i$ . Therefore the path of the ray on the two sides of A is, in general, not symmetrical. The necessary condition for the symmetry of the path around some point, in a spherically symmetrical atmosphere, is that the angle of incidence at this point from either side of the ray is  $\frac{\pi}{2}$ . This condition is satisfied at the point where  $\frac{dr}{d\phi}$  is zero.

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